Toric geometry of exceptional holonomy manifolds

Workshop on multisymplectic geometry,
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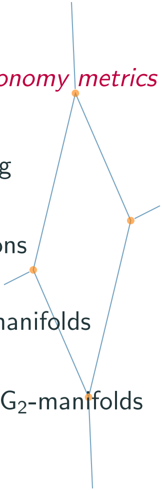
Outline

Ricci-flat special holonomy metrics with torus symmetry



Motivation: symplectic setting

Multi-Hamiltonian torus actions

Toric exceptional holonomy manifolds

Toric Asymptotically Conical G_2-manifolds

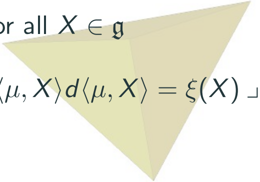
Motivation: symplectic setting

Toric symplectic geometry

(M^{2n}, ω) compact symplectic with effective Hamiltonian action of $G = T^n$. So have associated **moment map**

$$\mu\colon M \to \mathfrak{g}^* \cong \mathbb{R}^n$$

which is invariant and for all $X \in \mathfrak{g}$



$$\langle \mu, X \rangle d\langle \mu, X \rangle = \xi(X) \lrcorner\, \omega.$$

- If $b_1(M) = 0$, then T^n a action preserving ω is Hamiltonian iff all orbits are isotropic.
- codim of generic orbit equals that of target space of μ.
- Stabiliser of any point is subtorus of dim $n - \operatorname{rank} d\mu$.
- μ identifies orbit space, M/G, with a convex polytope.

HyperKähler

(M, g, I_1, I_2, I_3) is **hyperKähler** if each (g, I_ℓ) is a Kähler structure and $I_i I_j = I_k = -I_j I_i$, $(ijk) = (123)$; each $\omega_\ell = g(I_\ell \cdot, \cdot)$ is then symplectic.

Given $p \in M$,

$$\mathrm{Stab}_{\mathrm{GL}(T_pM)}(\omega_1, \omega_2, \omega_3) \cong \mathrm{Sp}(n) \leqslant \mathrm{SO}(4n).$$

As a consequence g has holonomy in $\mathrm{Sp}(n)$ and is Ricci-flat.

M has 2-sphere worth of symplectic forms, but g is Ricci-flat...

If M is compact any Killing vector field is parallel,
implying that holonomy of g reduces.

We are interested in torus symmetry, so take M to be non-compact.

Toric hyperKähler

Hypertoric is complete hyperKähler M^{4n} with effective tri-Hamiltonian $G = T^n$ action: this means we have hyperKähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) \colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^* \cong \mathbb{R}^{3n},$$

i.e., μ_ℓ is symplectic moment map for ω_ℓ.

- If $b_1(M) = 0$, then a T^n action preserving each ω_ℓ is tri-Hamiltonian iff all orbits are isotropic for each ω_ℓ.
- codim of generic orbit is $3n$, same as that of target space of μ.
- Stabiliser of any point is subtorus of dim $n - \frac{1}{3}\operatorname{rank} d\mu$.
- μ induces homeomorphism $M/T^n \to \mathbb{R}^{3n}$.

Locally, g is given by Gibbons-Hawking type ansatz:

$$g = \frac{1}{\det(V)} \theta^t \operatorname{adj}(V) \theta + \sum_{\ell=1}^{3} d\mu_\ell^t V d\mu_\ell,$$

θ connection 1-form and $V = (g(U_i, U_j))^{-1}$, with U_ℓ generating the torus action; V pos. def. sym. matrix of polyharmonic functions.

Ricci-flat special holonomy

In addition to hyperKähler, there are 3 other types of Ricci-flat geomtries appearing on Berger's holonomy list:

name	hol	dim	form deg
Calabi-Yau	$SU(n)$	$2n$	$2, n, n$
HyperKähler	$Sp(n)$	$4n$	$2, 2, 2$
G_2-mnfld	G_2	7	$3, 4$
Spin(7)-mnfld	Spin(7)	8	4

We have seen that for geometries defined by symplectic forms and admitting torus symmetry, moment map techniques can be used to construct many examples and obtain classifications.

What about the cases with higher degree closed forms?

Multi-Hamiltonian torus actions

Multi-Hamiltonian actions

M with closed $\alpha \in \Omega^{r+1}(M)$ preserved by action of Abelian G.

Action is **multi-Hamiltonian** if there is invariant $\nu\colon M \to \Lambda^r \mathfrak{g}^*$ s.t. $\forall X_i \in \mathfrak{g}$

$$\langle \nu, X_1 \wedge \cdots \wedge X_r \rangle d\langle \nu, X_1 \wedge \cdots \wedge X_r \rangle = \alpha(\xi(X_1), \ldots, \xi(X_r), \cdot\,).$$

Our interest is $G = T^n$, acting effectively:

- should take $n \geqslant r$;
- if $b_1(M) = 0$, then T^n action preser α is multi-Hamiltonian iff α pulls back to zero on each orbit.

If we have several closed invariant forms $\alpha_i \in \Omega^{r_i+1}(M)$ with multi-moment maps ν_i, we form the product multi-moment map

$$\nu = (\nu_1, \ldots, \nu_k)\colon\ M \to \bigoplus_{i=1}^{k} \Lambda^{r_i} \mathfrak{g}^*$$

Capturing orbit space with multi-moment maps

Let $M_0 \subset M$ be the open dense set where the torus G acts freely and let $q = \dim(M_0/G)$ be the codimension of generic orbits.

An interesting case is when the multi-moment map

$$\nu\colon\ M_0 \to \mathbb{R}^q$$

has full rank. Then ν locally exhibits M_0 as a principal G-bundle over $\mathcal{U} = \nu(M_0) \subset \mathbb{R}^q$.

For the Ricci-flat special holonomy geometries, the above requires:

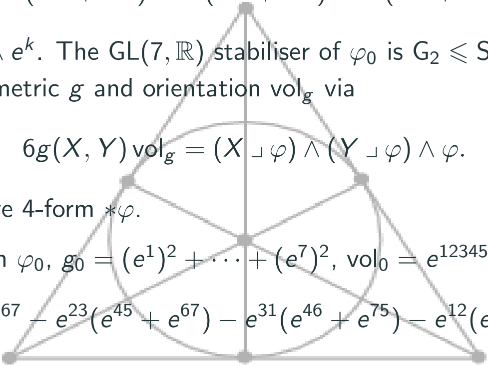
name	$\dim(M)$	$\deg \alpha_i$	G	q
Calabi-Yau	$2n$	$2, n, n$	T^{n-1}	$n+1$
HyperKähler	$4n$	$2, 2, 2$	T^n	$3n$
G_2	7	$3, 4$	T^3	4
Spin(7)	8	4	T^4	4

Toric exceptional holonomy manifolds

G_2-structures

M^7 with $\varphi \in \Omega^3(M)$ pointwise linearly equivalent to

$$\varphi_0 = e^{123} - e^1(e^{45} + e^{67}) - e^2(e^{46} + e^{75}) - e^3(e^{47} + e^{56}) \in \Lambda^3(\mathbb{R}^7)^*$$



$e^{ijk} = e^i \wedge e^j \wedge e^k$. The $\mathrm{GL}(7, \mathbb{R})$ stabiliser of φ_0 is $\mathrm{G}_2 \leqslant \mathrm{SO}(7)$.
It determines metric g and orientation vol_g via

$$6g(X, Y)\,\mathrm{vol}_g = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi.$$

So we also have 4-form $*\varphi$.

For model form φ_0, $g_0 = (e^1)^2 + \cdots + (e^7)^2$, $\mathrm{vol}_0 = e^{1234567}$ and

$$*\varphi_0 = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56}).$$

Holonomy of g is in G_2 iff $d\varphi = 0$ and $d * \varphi = 0$.

Spin(7)-structures

This geometry is defined in 8 dimensions by $\Phi \in \Omega^4(M^8)$ pointwise linearly equivalent to

$$\Phi_0 = e^0 \wedge \varphi_0 + *_{\varphi_0}\varphi_0 \in \Lambda^4(\mathbb{R}^8)^*;$$

$\mathrm{GL}(8,\mathbb{R})$ stabiliser of Φ_0 is $\mathrm{Spin}(7) \leqslant \mathrm{SO}(8)$.

Again, Φ determines metric g and volume form.

Holonomy of g is in Spin(7) iff $d\Phi = 0$.

Full holonomy examples with torus symmetry

As before, Ricci-flatness implies that full holonomy examples with torus symmetry must be non-compact.

The first complete examples were constructed 30+ years ago [Bryant-Salamon 1989]:

M	$\Lambda^2_-(S^4)$	$\Lambda^2_-(\mathbb{C}P^2)$	$\mathbb{S}(S^3)$	$\Sigma_-(S^4)$
Isom_0	$\mathrm{Sp}(2)$	$\mathrm{SU}(3)$	$\mathrm{SU}(2)^3$	$\mathrm{Sp}(2)\times\mathrm{SU}(2)$
$\mathrm{rank}(\mathrm{Isom})$	2	2	3	3

Above list already provide examples with full holonomy admitting effective torus action.

Multi-Hamiltonian G_2- and $\mathrm{Spin}(7)$-manifolds

Have anticipated that, from toric viewpoint, most interesting cases should be G_2-manifolds with T^3-symmetry and $\mathrm{Spin}(7)$-manifolds with T^4-symmetry.

Other situations with torus symmetry that have been investigated previously include:

- [Madsen-Swann '12] explored G_2-manifolds with T^2-symmetry, multi-Hamiltonian for φ; here $\nu\colon M \to \mathbb{R}$ whilst $\dim(M_0/T^2) = 5$;
- [Baraglia '10] studied G_2-manifolds with T^4-symmetry, multi-Hamiltonian for φ. Then $\nu\colon M \to \mathbb{R}^6$, but $\dim(M_0/T^4) = 3$;
- [Madsen '11] described $\mathrm{Spin}(7)$-manifolds with multi-Hamiltonian T^3-symmetry. So $\nu\colon M \to \mathbb{R}$ whilst $\dim(M_0/T^3) = 5$.

Toric G_2-manifolds: verifying expectations

Consider a G_2-manifold (M, φ) with effective T^3 action that is multi-Hamiltonian for both φ and $*\varphi$.

Let U_1, U_2, U_3 generate the torus action. So $\varphi(U_1, U_2, U_3) = 0$ and multi-moment map $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu) \colon M \to \mathbb{R}^4$ satisfies

$$d\nu_i = U_j \wedge U_k \lrcorner \varphi \qquad (ijk) = (123)$$
$$d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi.$$

Recall that, at p, we can write
$\varphi = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356}$,
$*\varphi = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56})$.

Moreover, for $p \in M_0$, we can choose our G_2-basis s.t. $\mathrm{Span}\{U_1, U_2, U_3\} = \mathrm{Span}\{E_5, E_6, E_7\}$.

Hence, $(\nu, \mu) \colon M_0 \to \mathbb{R}^4$ has full rank and multi-moment map locally exhibits M_0 as principal T^3-bundle over $\mathcal{U} \subset \mathbb{R}^4$.

Toric Spin(7)-manifolds: verifying expectations

Similarly, consider a Spin(7)-manifold (M, Φ) with an effective multi-Hamiltonian T^4 action.

Let $U_0, \ldots, U_3$ be generators of the torus action. Then $\Phi(U_0, U_1, U_2, U_3) = 0$ and multi-moment map $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)\colon M \to \mathbb{R}^4$ is chosen to satisfy

$$d\nu_i = (-1)^i U_j \wedge U_k \wedge U_\ell \lrcorner \Phi \quad (ijk\ell) = (0123).$$

This time, at p, we have $\Phi = e^0 \wedge \varphi_0 + *_{\varphi_0}\varphi_0$, and for $p \in M_0$ we may take our Spin(7)-basis s.t. $\mathrm{Span}\{U_i\} = \mathrm{Span}\{E_0, E_5, E_6, E_7\}$.

As before, it follows that $\nu\colon M_0 \to \mathbb{R}^4$ has full rank and so locally realises M_0 as a principal T^4-bundle over $\mathcal{U} \subset \mathbb{R}^4$.

Where action is free: toric G_2 and $\mathrm{Spin}(7)$

We have that M_0 is the total space of a principal T^n-bundle, $n = 3, 4$, with connection 1-forms $\theta_i \in \Omega^1(M_0)$ that satisfy

$$\theta_i(U_j) = \delta_{ij}, \quad \theta_i(X) = 0 \quad \forall X \perp \mathrm{Span}\{U_i\}.$$

On M_0 we can define a positive definite symmetric $n \times n$-matrix of functions by:

$$V = (g(U_i, U_j))^{-1}.$$

This enables us to write down a toric G_2-structure in a way resembling what we had for hypertoric case:

$$g = \tfrac{1}{\det V}\theta^t \operatorname{adj}(V)\theta + d\nu^t \operatorname{adj}(V) d\nu + \det(V) d\mu^2$$

$$\varphi = -\det(V) d\nu_{123} + d\mu \wedge d\nu^t \operatorname{adj}(V)\theta + \underset{ijk}{\mathfrak{S}}\, \theta_{ij} \wedge d\nu_k$$

$$*\varphi = \theta_{123} d\mu + \tfrac{1}{2\det(V)}\big(d\nu^t \operatorname{adj}(V)\theta\big)^2 + \det(V) d\mu \wedge \underset{ijk}{\mathfrak{S}}\, \theta_i \wedge d\nu_{jk}$$

For toric Spin(7)-manifolds, we have:

$$g = \tfrac{1}{\det(V)}\theta^t \operatorname{adj}(V)\theta + d\nu^t \operatorname{adj}(V) d\nu$$

$$\Phi = \det(V) \mathop{\mathfrak{S}}_{ijk\ell}(-1)^i\theta_i \wedge d\nu_{jk\ell} + \mathop{\mathfrak{S}}_{ijk\ell}(-1)^\ell\theta_{ijk} \wedge d\nu_\ell + \tfrac{1}{2\det(V)}(d\nu^t \operatorname{adj}(V)\theta)^2.$$

Note that G_2- and Spin(7)-structures defined by the above formulae are generally not torsion-free, so holonomy reduction is not guaranteed.

Torsion-free condition amounts to following system of PDEs:
$V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^n))$, $n = 3, 4$, is a positive definite solution to

$$\sum_i \frac{\partial V_{ij}}{\partial \nu_i} = 0 \quad \text{for each } j \qquad \text{(divergence-free)}$$

and

$$L(V) + Q(dV) = 0 \qquad \text{(elliptic)}$$

where

$$\underbrace{L = \frac{\partial^2}{\partial \mu^2} + \sum_{ij} V_{ij} \frac{\partial^2}{\partial \nu_i \partial \nu_j},}_{\mathrm{G}_2} \qquad \underbrace{L = \sum_{ij} V_{ij} \frac{\partial^2}{\partial \nu_i \partial \nu_j}}_{\mathrm{Spin}(7)}$$

and

$$Q(dV)_{ij} = -\sum_{ab} \frac{\partial V_{ia}}{\partial \nu_b} \frac{\partial V_{bj}}{\partial \nu_a}$$

Naturality: L and Q are preserved, up to scale, by $\mathrm{GL}(n, \mathbb{R})$ change of basis, and this specifies Q uniquely.

Diagonal solutions: examples of incomplete toric G_2

$V = \mathrm{diag}(V_1, V_2, V_3)$ (divergence-free) and off-diagonal terms in (elliptic) read

$$\frac{\partial V_i}{\partial \nu_i} = 0, \quad \frac{\partial V_i}{\partial \nu_j}\frac{\partial V_j}{\partial \nu_i} = 0 \quad (i \neq j)$$

Either $V = \mathrm{diag}(V_1(\nu_2, \mu), V_2(\nu_3, \mu), V_3(\nu_1, \mu))$, linear in each variable.
E.g. $V = \mu 1_3$, $\mu > 0$, full holonomy G_2:

$$g = \frac{1}{\mu}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2(d\nu_1^2 + d\nu_2^2 + d\nu_3^2) + \mu^3 d\mu^2$$

$$d\theta_i = d\nu_j \wedge d\nu_k \quad (ijk) = (123).$$

Or get elliptic hierachy $V_3 = V_3(\mu)$, $V_2(\nu_3, \mu)$, $V_1 = V_1(\nu_2, \nu_3, \mu)$:

$$\frac{\partial^2 V_3}{\partial \mu^2} = 0, \quad \frac{\partial^2 V_2}{\partial \mu^2} + V_3\frac{\partial^2 V_2}{\partial \nu_3^2} = 0, \quad \frac{\partial^2 V_1}{\partial \mu^2} + V_2\frac{\partial^2 V_1}{\partial \nu_2^2} + V_3\frac{\partial^2 V_1}{\partial \nu_3^2} = 0$$

E.g. $V_1 = 2\mu^5 - 15\mu^2\nu_3^2 - 5\nu_2^2$, $V_2 = \mu^3 - 3\nu_3^2$, $V_3 = \mu$.

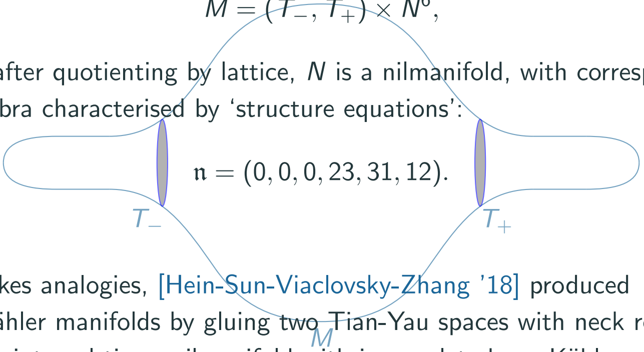
Wishful thinking: incomplete examples as necks

Underlying manifold in first example above is of the form

$$M = (T_-, T_+) \times N^6,$$

where, after quotienting by lattice, N is a nilmanifold, with corresponding Lie algebra characterised by 'structure equations':

$$\mathfrak{n} = (0, 0, 0, 23, 31, 12).$$



If one likes analogies, [Hein-Sun-Viaclovsky-Zhang '18] produced hyperKähler manifolds by gluing two Tian-Yau spaces with neck region given by interval times nilmanifold with incomplete hyperKähler metric...

Might it be possible to to produce compact G_2-manifolds, using incomplete toric gluing blocks as neck?

Toric $\mathrm{Spin}(7)$: first examples with rank 4 symmetry

$V = (V_0, V_1, V_2, V_3)$ with (divergence-free) and off-diagonal terms in (elliptic) similar to G_2 case.

Again one option is a linear solution. Simplest full holonomy of the form $V_i = \nu_{i+1}$, $\nu_i > 0$, $i \in \mathbb{Z}_4$:

$$g = \frac{1}{\nu_1}\theta_0^2 + \frac{1}{\nu_2}\theta_1^2 + \frac{1}{\nu_3}\theta_2^2 + \frac{1}{\nu_0}\theta_3^2 + \nu_2\nu_3\nu_0 d\nu_0^2 + \nu_1\nu_3\nu_0 d\nu_1^2 + \nu_1\nu_2\nu_0 d\nu_2^2 + \nu_1\nu_2\nu_3 d\nu_3^2,$$

$$d\theta_0 = -\nu_2 d\nu_{23},\ d\theta_1 = \nu_3 d\nu_{30},\ d\theta_2 = -\nu_0 d\nu_{01},\ d\theta_3 = \nu_1 d\nu_{12}.$$

Otherwise, get elliptic hierachy. Full holonomy arises, e.g., from taking $V_0 = V_0(\nu_1, \nu_2)$, $V_1 = V_1(\nu_2, \nu_3)$, $V_2 = \nu_3$, $V_3 = \nu_0$ with

$$V_1 \frac{\partial^2 V_0}{\partial \nu_1^2} + \nu_3 \frac{\partial^2 V_0}{\partial \nu_2^2} = 0, \quad \nu_3 \frac{\partial^2 V_1}{\partial \nu_2^2} + \nu_0 \frac{\partial^2 V_1}{\partial \nu_3^2} = 0.$$

Towards global picutre: special orbits

For toric G_2-manifolds, we cannot have fixed points as isotropy group acts faithfully as subgroup of G_2 on normal space of the orbit.
In addition, it turns out stabilisers are connected:

Proposition

For toric G_2-manifolds every non-trivial stabiliser of the T^3 action is subtorus of rank $\leqslant 2$.

For toric Spin(7) similar conclusion as there is always one isotropy invariant direction, forcing isotropy group to be a subgroup of $\mathrm{G}_2 \leqslant \mathrm{Spin}(7)$.

As a consequence most of the hard work in understanding behaviour around points with $G_p \neq \{e\}$ amounts to understanding G_2-case.

Flat models - toric G_2

For stabiliser S^1, the flat model is

- $M = T^2 \times \mathbb{R} \times \mathbb{C}^2$
- $G = T^2 \times S^1 = T^2 \times \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$

and topologically

$$M/G = (T^2 \times \mathbb{R} \times \mathbb{C}^2)/(T^2 \times S^1) = \mathbb{R} \times (\mathbb{C}^2/S^1) = \mathbb{R} \times C(S^3/S^1) = \mathbb{R}^4.$$

For the case of stabiliser T^2, associated flat model is

- $M = S^1 \times \mathbb{C}^3$
- $G = S^1 \times T^2 = S^1 \times \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \colon \theta_1 + \theta_2 + \theta_3 = 0\}$

Topologically, we then have

$$M/G = (S^1 \times \mathbb{C}^3)/(S^1 \times T^2) = \mathbb{C}^3/T^2 = C(S^5)/T^2 = C(S^5/T^2) = \mathbb{R}^4.$$

Hence for toric G_2-manifolds orbit space is topological manifold. Same statement holds for toric Spin(7)-manifolds.

Multi-moment map for flat models

For $M = S^1 \times \mathbb{C}^3$, so T^2 stabiliser, we have

$$\varphi = \tfrac{i}{2} dx \wedge (dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$$
$$*\varphi = \mathrm{Im}(dz_{123}) \wedge dx - \tfrac{1}{8}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}})^2$$

with T^3 generated by

$$U_1 = \frac{\partial}{\partial x}, \quad U_k = 2\,\mathrm{Re}\left(i\left(z_{k-1}\frac{\partial}{\partial z_{k-1}} - z_3\frac{\partial}{\partial z_3}\right)\right) \quad k = 2, 3.$$

Associated multi-moment map $(\nu, \mu)\colon M \to \mathbb{R}^4$ is

$$\nu_1 + i\mu = -\overline{z_1 z_2 z_3}, \quad \nu_2 = \tfrac{1}{2}(|z_2|^2 - |z_3|^2), \quad \nu_3 = -\tfrac{1}{2}(|z_1|^2 - |z_3|^2)$$

As for hypertoric manifolds, analysis of this special case gives:

Proposition

(ν, μ) induces a homeomorphism $M/G = \mathbb{C}^3/T^2 \to \mathbb{R}^4$.

Similar conclusion holds for S^1 stabiliser.

General case: global local coordinates

Theorem

For toric G_2- and $\mathrm{Spin}(7)$-manifolds the multi-moment map induces a local homeomorphism $M/G \to \mathbb{R}^4$.

Key steps in proof:

- properties of commuting Killing fields at zeros;
- approximation by the flat model;
- 'controlled comparison' with flat model to infer injectivity around singular orbit; this gets quite involved for the case of T^2 stabiliser.

Approximation by flat model - toric G_2

Consider the case of having stabiliser T^2 at p. At this point, we can ensure φ and $*\varphi$ agree with flat model. From study of commuting Killing fields we know that, at p, it can be assumed that

$$U_2 = 0 = U_3, \quad \nabla U_1 = 0, \quad \nabla^2 U_2 = 0 = \nabla^2 U_3,$$

and U_1, ∇U_2, ∇U_3 agree with flat model.

Then, using $\nabla\varphi = 0$ we get:

$$\nabla^\ell \nu_i = \sum_{q+r=\ell-1} \binom{\ell-1}{q} \varphi(\nabla^q U_j, \nabla^r U_k, \cdot) \quad (ijk) = (123)$$

with similar explicit expressions for $\nabla^\ell \mu$, obtained using $\nabla *\varphi = 0$.

Lemma

At p, ν_1, μ agree with the flat model to order 4 and ν_2, ν_3 agree with the flat model to order 3.

For S^1 stabiliser, we obtain that $\nabla^\ell \nu_1$, $\nabla^\ell \nu_2$ and $\nabla^\ell \mu$ agree with flat model to order 2.

Combinatorial data: image of singular locus

Recall that for toric G_2, we have:

$$d\nu_i = \varphi(U_j, U_k, \cdot\,) \quad (ijk) = (123),$$
$$d\mu = *\varphi(U_1, U_2, U_3, \cdot\,).$$

So if, say, U_3 vanishes on a collection of singular orbits, then ν_1, ν_2 and μ are constant on that collection and we get a line segment parameterised by ν_3.

Inspecting the flat models, we get the following in general:

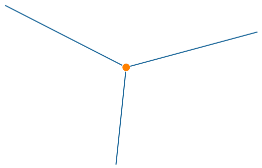
- S^1 stabilisers $\mapsto$ lines in $\mathbb{R}^3 \times \{\mu\} \subset \mathbb{R}^4$ of rational slope;
- T^2 stabiliser $\mapsto$ a point in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$;
- any intersection is triple with primitive slope vectors summing to zero.

Hence, we get a collection of trivalent graphs in $\mathbb{R}^3 \times \mathbb{R}$, each connected component contained in some $\mathbb{R}^3 \times \{\mu\} \subset \mathbb{R}^4$.

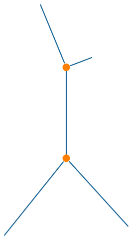
For toric Spin(7) a similar conclusion holds, but with no distinguished direction in target $\mathbb{R}^4$.

Trivalent graphs for some known toric G_2-manifolds

Flat model $S^1 \times \mathbb{C}^3$:



Bryant-Salamon example(s) on $\mathbb{S}(S^3)$:



For the above examples, multi-moment map induces homeomorphism $M/T^3 \to \mathbb{R}^4$.

Some key questions to be addressed

- I have not given you any examples complete toric Spin(7)-manifolds with full holonomy: do such examples exist?
- For g complete, can we show that the multi-moment map furnishes a homeomorphism $M/G \cong \mathbb{R}^4$? Maybe, we have to impose additional assumptions on asymptotic behaviour of metric.
- How do combinatorial data fit into classification scheme? What trivalent graphs correspond to complete G_2-manifolds?

Toric Asymptotically Conical G_2-manifolds

Toric Calabi-Yau 3-folds

Consider a Calabi-Yau 3-fold (N^6, ω, Ψ) with an effective $G = T^2$ action, multi-Hamiltonian for ω and $\operatorname{Re}\Psi$, $\operatorname{Im}\Psi$.

The product $S^1 \times N$ is toric G_2 in a rather trivial way. In particular, we deduce that the multi-moment map $N \to \mathbb{R}^4$ induces a homeomorphism $N/G \to \mathbb{R}^4$.

Toric Calabi-Yau 3-folds, as defined traditionally, come with a T^3 action which is Hamiltonian for ω in the usual sense, but does not preserve Ψ.

There is always $G = T^2 \leqslant T^3$ which acts multi-Hamiltonian in above sense for (ω, Ψ).

We are particularly interested in toric Calabi-Yau 3-folds that are asymptotic to the Riemannian cone $C(\Sigma)$ over a Sasaki-Einstein 5-manifold Σ.

Towards classification results

Starting from a toric asymptotically conical Calabi-Yau 3-fold (N, ω, Ψ), one looks for a non-trivial circle bundle $M \to N$ such that

$$c_1(M) \cup [\omega] = 0 \in H^4(N)$$

[Foscolo-Haskins-Nordström '17] then guarantee the existence of a 1-paramenter family φ_ε, $\varepsilon \in (0, \varepsilon_0)$, of asymptotically locally conical G_2-structures on M.

The G_2-manifolds constructed in this way are all toric.

Toric asymptotically conical Calabi-Yau 3-folds well studied, so first classification results *might* be feasible and this is work in progress with Kael Dixon and Simon Salamon.